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## Random Fixed Point Theorems with an Application to Random Differential Equations in Banach Spaces

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### INTRODUCTION

Random fixed point theorems are stochastic generalizations of (classical) fixed point theorems, and are required for the theory of random equations, just as in the theory of deterministic equations, (classical) fixed point theorems are of fundamental importance. In Polish spaces, i.e., separable complete metric spaces, random fixed point theorems for contraction mappings were proved by Špaček [34] and Hanš [12, 13, 14], etc. Mukherjea [cf. Bharucha–Reid [3, p. 111]] gave a random fixed point theorem of Schauder type on an atomic probability measure space. Then Prakasa Rao [32] extended this result and obtained a theorem of Krasnosel'skii type on a same measure space. Recently, Bharucha–Reid [2] generalized results of Mukherjea and Prakasa Rao to the cases on general probability measure spaces. We refer to Bharucha–Reid [3] for a survey of related results.

In [18] we obtained a random fixed point theorem for a multivalued contraction mapping in a Polish space. In this paper, in Section 2 we give several random fixed point theorems for various singlevalued mappings, e.g., condensing or nonexpansive mappings, etc., in Banach or Hilbert spaces on general measurable spaces. Among them, we have results of Mukherjea, Prakasa Rao and Bharucha–Reid on measurable spaces. Then, in Section 3 we prove corresponding theorems for multivalued mappings. In Section 4, as an application we show the existence of a random solution of a differential equation in a Banach space by using Theorem 2.1. To prove them, some results on measurability and measurable selectors of multivalued mappings given by Kuratowski and Ryll–Nardzewski [23] and Himmelberg [16] play crucial roles.

### 1. PRELIMINARIES

Throughout this paper,  $(\Omega, \mathcal{A})$  denotes a measurable space. Let  $X$  be a metric space with the metric  $d$ . Let  $2^X$  be the family of all subsets of  $X$ ,  $CD(X)$  all nonempty closed subsets of  $X$ ,  $CB(X)$  all nonempty bounded closed subsets of

$X, K(X)$  all nonempty compact subsets of  $X$ , respectively. A mapping  $F: \Omega \rightarrow 2^X$  is called ( $\mathcal{A}$ -)measurable if for any open subset  $B$  of  $X$ ,  $F^{-1}(B) = \{\omega \in \Omega: F(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$ . This type of measurability is usually called weakly measurable (cf. Himmelberg [16]), but in this paper we always use this type of measurability, thus we omit the term “weakly” for simplicity. Notice that when  $F(\omega) \in K(X)$  for all  $\omega \in \Omega$ , then  $F$  is measurable if and only if  $F^{-1}(C) \in \mathcal{A}$  for every closed subset  $C$  of  $X$  (cf. Himmelberg [16]). A measurable mapping  $\xi: \Omega \rightarrow X$  is called a *measurable selector* of a measurable mapping  $F: \Omega \rightarrow CD(X)$  if  $\xi(\omega) \in F(\omega)$  for each  $\omega \in \Omega$ . A mapping  $f: \Omega \times X \rightarrow X$  is called a *random operator* if for any  $x \in X$ ,  $f(\cdot, x)$  is measurable. Similarly, a mapping  $F: \Omega \times X \rightarrow CD(X)$  is a *random operator* if for every  $x \in X$ ,  $F(\cdot, x)$  is measurable. A measurable mapping  $\xi: \Omega \rightarrow X$  is called a *random fixed point* of a random operator  $f: \Omega \times X \rightarrow X$  (or  $F: \Omega \times X \rightarrow CD(X)$ ) if for every  $\omega \in \Omega$ ,  $f(\omega, \xi(\omega)) = \xi(\omega)$  (or  $\xi(\omega) \in F(\omega, \xi(\omega))$ ).

A mapping  $f: X \rightarrow X$  is called *compact* if  $f$  is continuous and  $f(X)$  is precompact.  $f$  is called *asymptotically regular* if for any  $x \in X$ ,  $d(f^n(x), f^{n+1}(x)) \rightarrow 0$  as  $n \rightarrow \infty$ .  $f$  is called *k-Lipschitz*, where  $k \geq 0$ , if for every  $x, y \in X$ ,  $d(f(x), f(y)) \leq kd(x, y)$ . A *k-Lipschitz* mapping  $f$  is a *k-contraction* or a *nonexpansive* mapping if  $k < 1$  or  $k = 1$  respectively.  $f$  is called *k-set-Lipschitz* ( $k \geq 0$ ) if  $f$  is continuous and for any bounded subset  $C$  of  $X$ ,  $\gamma(f(C)) \leq k\gamma(C)$ , where  $\gamma(B) = \inf\{c > 0: B \text{ can be covered by a finite number of sets of diameter } \leq c\}$ .  $\gamma(B)$  is called the (*set*-)measure of noncompactness of  $B$ . This is originally due to Kuratowski (cf. [22]). We also refer to Furi and Vignoli [10, 11] and Nussbaum [27]. A *k-set-Lipschitz* mapping  $f$  is a *k-set-contraction* if  $k < 1$ .  $f$  is called (*set*-)condensing if  $f$  is continuous and for each bounded subset  $C$  of  $X$  with  $\gamma(C) > 0$ ,  $\gamma(f(C)) < \gamma(C)$ . It is clear that a *k-set-contraction* mapping is condensing. If  $X$  is a subset of a Banach space  $E$ , then a mapping  $f: X \rightarrow X$  of the form  $f = g + h$ , where  $g: X \rightarrow E$  is a *k-contraction* and  $h: X \rightarrow E$  is compact, is a *k-set-contraction*.

A mapping  $F: X \rightarrow CD(X)$  is called *compact* if  $F(X) = \bigcup_{x \in X} F(x)$  is precompact.  $F$  is called *asymptotically regular* if for each  $x \in X$ , there exists a sequence  $\{x_n\}$  such that  $x_0 = x$ ,  $x_{n+1} \in F(x_n)$  and  $d(x_n, x_{n+1}) \rightarrow 0$  as  $n \rightarrow \infty$ .  $F$  is called *upper (lower) semicontinuous* if for any closed (open) subset  $C$  of  $X$ ,  $F^{-1}(C)$  is closed (open).  $F$  is called *continuous* if  $F$  is both upper and lower semicontinuous.  $F: X \rightarrow CB(X)$  is called *k-Lipschitz* ( $k \geq 0$ ) if for any  $x, y \in X$ ,  $D(F(x), F(y)) \leq kd(x, y)$ , where  $D$  is the Hausdorff metric on  $CB(X)$  induced by the metric  $d$ . If  $k < 1$  or  $k = 1$ , then  $F$  is called a *k-contraction* or a *nonexpansive* mapping, respectively. Notice that when  $F(x) \in K(X)$  for all  $x \in X$ , then  $F$  is continuous if and only if  $F$  is continuous from  $X$  into the metric space  $(K(X), D)$ , that is, for each  $x_0 \in X$ , given  $p > 0$ , there exists a  $q > 0$  such that  $D(F(x), F(x_0)) < p$  whenever  $d(x, x_0) < q$  (cf. Castaing [8, Théorème 4.1]).  $F$  is called (*set*-)condensing if for any bounded subset  $C$  of  $X$  with  $\gamma(C) > 0$ ,  $\gamma(F(C)) < \gamma(C)$ .

A random operator  $f: \Omega \times X \rightarrow X$  is called *continuous (compact, etc.)* if for each  $\omega \in \Omega$ ,  $f(\omega, \cdot)$  is continuous (compact, etc.). Similarly, a random operator  $F: \Omega \times X \rightarrow CD(X)$  is called *continuous (compact, etc.)* if for any  $\omega \in \Omega$ ,  $F(\omega, \cdot)$  is continuous (compact, etc.).

If  $X$  is a subset of a Banach space, then let  $CK(X)$  be the family of all non-empty compact convex subsets of  $X$ ,  $WK(X)$  all nonempty weakly compact subsets of  $X$ , respectively. A mapping  $F: \Omega \rightarrow 2^X$  is called *w-measurable* if for every weakly closed subset  $C$  of  $X$ ,  $F^{-1}(C) \in \mathcal{A}$ .  $X$  is said to be *starshaped* if there exists a point  $v \in X$  such that for any  $x \in X$  and  $k$  ( $0 < k < 1$ ),  $kv + (1 - k)x \in X$ .  $v$  is called a *starcenter* of  $X$ .

## 2. RANDOM FIXED POINT THEOREMS FOR SINGLEVALUED MAPPINGS

First, we prove a random fixed point theorem for a condensing mapping. A fixed point theorem for a condensing mapping was given by Furi and Vignoli [11].

**THEOREM 2.1.** *Let  $X$  be a separable closed convex subset of a Banach space,  $f: \Omega \times X \rightarrow X$  a condensing random operator. Suppose that for any  $\omega \in \Omega$ ,  $f(\omega, X)$  is bounded. Then there exists a random fixed point  $\xi: \Omega \rightarrow X$  of  $f$ .*

*Proof.* Take a countable dense subset  $\{x_n\}$  of  $X$ . Define a mapping  $F: \Omega \rightarrow 2^X$  by  $F(\omega) = \{x \in X: f(\omega, x) = x\}$ , then by a fixed point theorem of Furi and Vignoli [11],  $F(\omega) \in K(X)$  for all  $\omega \in \Omega$ . We show that  $F$  is measurable. For any nonempty closed subset  $C$  of  $X$ , denote

$$L(C) = \bigcap_{n=1}^{\infty} \bigcup_{x_i \in C_n} \{\omega \in \Omega: \|f(\omega, x_i) - x_i\| < 2/n\},$$

where  $C_n = \{x \in X: d(x, C) < 1/n\}$  and  $d(x, C) = \inf\{\|x - y\|: y \in C\}$ . Then we obtain  $F^{-1}(C) = L(C)$ . Indeed, it is easy to see that  $F^{-1}(C) \subset L(C)$ . Conversely, if  $\omega \in L(C)$ , then for each  $n$ , there exists  $x_{i(n)}$  such that  $d(x_{i(n)}, C) < 1/n$  and  $\|f(\omega, x_{i(n)}) - x_{i(n)}\| < 2/n$ . The subset  $B = \{x_{i(n)}: n = 1, 2, \dots\}$  of  $X$  is precompact by Furi and Vignoli [10, Theorem 1]. Thus, there is a subsequence of  $B$  converging to some  $y_0 \in X$ . We have  $y_0 \in C$  and  $f(\omega, y_0) = y_0$ , showing  $\omega \in F^{-1}(C)$ . Hence,  $F^{-1}(C) = L(C)$  and  $F$  is measurable. By Kuratowski and Ryll-Nardzewski [23], there exists a measurable selector  $\xi: \Omega \rightarrow X$  of  $F$ . This  $\xi$  is the desired random fixed point of  $f$ . Q.E.D.

As a corollary, we have the following random fixed point theorem of Schauder type. If  $X$  is compact, this was essentially obtained by Bharucha-Reid [2].

**COROLLARY 2.2.** *Let  $X$  be a compact (or separable and closed) convex subset of a Banach space,  $f: \Omega \times X \rightarrow X$  a compact random operator. Then  $f$  has a random fixed point.*

**Remark 2.3.** If  $X$  is a separable, bounded and complete metric space and  $f: \Omega \times X \rightarrow X$  is a condensing random operator such that  $F(\omega)$  is nonempty for all  $\omega \in \Omega$ , then  $f$  has a random fixed point by the same method of the proof of Theorem 2.1.

When ranges are not necessarily bounded, then a random fixed point theorem of Krasnosel'skii type holds. Bharucha-Reid [2] proved a similar result in the case that  $X$  is compact and convex. For fixed point theorems of Krasnosel'skii type, we refer to Krasnosel'skii [21] and Reinermann [33].

**THEOREM 2.4.** *Let  $X$  be a closed convex subset of a separable Banach space  $E$ ,  $g: \Omega \times X \rightarrow E$  a random operator such that for any  $\omega \in \Omega$ ,  $g(\omega, \cdot)$  is a  $k(\omega)$ -contraction. Let  $h: \Omega \times X \rightarrow E$  be a compact random operator. Suppose that for any  $\omega \in \Omega$ ,  $g(\omega, x) + h(\omega, y) \in X$  whenever  $x, y \in X$ . Then  $g + h$  has a random fixed point.*

*Proof.* For each fixed  $y \in X$ , define a mapping  $T_y: \Omega \times X \rightarrow X$  by  $T_y(\omega, x) = g(\omega, x) + h(\omega, y)$ , then, since  $E$  is separable,  $T_y$  is a random operator (cf. Bharucha-Reid [3, p. 18]) such that for each  $\omega \in \Omega$ ,  $T_y(\omega, \cdot)$  is a  $k(\omega)$ -contraction. Thus, by Hanš [12], there exists a unique random fixed point  $\xi_y$  of  $T_y$ . Define a random operator  $f: \Omega \times X \rightarrow X$  by  $f(\omega, x) = \xi_x(\omega)$ , then  $f$  is continuous. In fact, if  $x, y \in X$ , then

$$\begin{aligned} \|h(\omega, x) - h(\omega, y)\| &= \|\xi_x(\omega) - g(\omega, \xi_x(\omega)) - \xi_y(\omega) + g(\omega, \xi_y(\omega))\| \\ &\geq \|\xi_x(\omega) - \xi_y(\omega)\| - \|g(\omega, \xi_x(\omega)) - g(\omega, \xi_y(\omega))\| \\ &\geq \|\xi_x(\omega) - \xi_y(\omega)\| - k(\omega) \|\xi_x(\omega) - \xi_y(\omega)\| \\ &= (1 - k(\omega)) \|f(\omega, x) - f(\omega, y)\|. \end{aligned}$$

Since  $h$  is compact,  $f$  is compact by the above inequality. Hence, by Corollary 2.2 there exists a random fixed point  $\xi$  of  $f$ . It is obvious that  $\xi$  is a random fixed point of  $g + h$ . Q.E.D.

The following is another random fixed point theorem of Krasnosel'skii type. A related fixed point theorem of this form was given by Reinermann [33]. Before we state the theorem, we recall a few definitions. A Banach space  $E$  is said to satisfy *Opial's condition* (cf. Opial [28] and Lami Dozo [24]) if the following holds: if  $\{x_n\}$  converges weakly to  $x_0$ , and  $x \neq x_0$ , then  $\liminf \|x_n - x\| > \liminf \|x_n - x_0\|$ . Banach spaces satisfying Opial's condition include Hilbert spaces and  $\ell^p$  ( $1 \leq p < \infty$ ) spaces. A mapping  $f$  of a subset  $X$  of a Banach space  $E$  into  $E$  is called *completely continuous* if  $\{x_n\}$  converges weakly to  $x_0$ , then  $\{f(x_n)\}$  converges to  $f(x_0)$ .  $f$  is called *demiclosed* if  $\{x_n\} \subset X$  is such that  $\{x_n\}$  converges weakly to  $x_0$  and  $\{f(x_n)\}$  converges to  $y_0$  in  $E$ , then  $f(x_0) = y_0$ .

**THEOREM 2.5.** *Let  $X$  be a weakly compact convex subset of a separable Banach space  $E$ ,  $g: \Omega \times X \rightarrow E$  a nonexpansive random operator,  $h: \Omega \times X \rightarrow E$  a completely continuous random operator. Suppose that  $E$  is uniformly convex or satisfies Opial's condition and for any  $\omega \in \Omega$ ,  $g(\omega, x) \neq h(\omega, x) \in X$  if  $x \in X$ . Then there exists a random fixed point of  $g + h$ .*

*Proof.* Take an element  $v \in X$  and a sequence  $\{k_n\}$  of real numbers such that  $0 < k_n < 1$  and  $k_n \rightarrow 0$  as  $n \rightarrow \infty$ . For each  $n$ , define a mapping  $f_n: \Omega \times X \rightarrow X$  by

$$f_n(\omega, x) = k_n v + (1 - k_n)(g(\omega, x) + h(\omega, x)),$$

then  $f_n$  is a  $(1 - k_n)$ -set-contraction random operator. Hence, by Theorem 2.1 there is a random fixed point  $\xi_n$  of  $f_n$ . For each  $n$ , define  $F_n: \Omega \rightarrow WK(X)$  by

$$F_n(\omega) = w\text{-cl}\{\xi_i(\omega): i \geq n\},$$

where  $w\text{-cl}(C)$  is the weak closure of  $C$ . Let  $F: \Omega \rightarrow WK(X)$  be a mapping defined by

$$F(\omega) = \bigcap_{n=1}^{\infty} F_n(\omega),$$

then, since the weak topology on  $X$  is a metric topology (cf. Dunford and Schwartz [9, p. 434]),  $F$  is  $w$ -measurable by Himmelberg [16, Theorem 4.1]. Thus, by Kuratowski and Ryll-Nardzewski [23], there is a  $w$ -measurable selector  $\xi$  of  $F$ . For any  $x^* \in E^*$  (the dual space of  $E$ ),  $x^*(\xi(\cdot))$  is measurable as a numerically-valued function on  $\Omega$ . Since  $E$  is separable,  $\xi$  is measurable (cf. Bharucha-Reid [3, pp. 14-16] and Hille and Phillips [15, p. 72]). We show that  $\xi$  is a random fixed point of  $g + h$ . Fix  $\omega \in \Omega$  arbitrarily, then some subsequence  $\{\xi_m(\omega)\}$  of  $\{\xi_n(\omega)\}$  converges weakly to  $\xi(\omega)$ . Hence,  $\{h(\omega, \xi_m(\omega))\}$  converges to  $h(\omega, \xi(\omega))$ . On the other hand, we have

$$\xi_m(\omega) - g(\omega, \xi_m(\omega)) = h(\omega, \xi_m(\omega)) + k_m\{v - g(\omega, \xi_m(\omega)) - h(\omega, \xi_m(\omega))\}.$$

Thus,  $\{\xi_m(\omega) - g(\omega, \xi_m(\omega))\}$  converges to  $h(\omega, \xi(\omega))$ . Since  $g(\omega, \cdot)$  is nonexpansive,  $I - g(\omega, \cdot)$  is demiclosed by Browder [7] if  $E$  is uniformly convex, or by Opial [28] if  $E$  satisfies Opial's condition, where  $I$  is the identity mapping of  $E$ . Therefore, it follows that  $\xi(\omega) - g(\omega, \xi(\omega)) = h(\omega, \xi(\omega))$ . Q.E.D.

In a Hilbert space, we can give more precise results for nonexpansive mappings as follows. If  $M$  is a nonempty closed convex subset of a Hilbert space  $H$ , we denote by  $P_M$  the (metric) projection of  $H$  onto  $M$ . It is well known that  $P_M$  is a nonexpansive mapping (cf. Phelps [31]).

**THEOREM 2.6.** *Let  $X$  be a bounded closed convex subset of a separable Hilbert space,  $f: \Omega \times X \rightarrow X$  a nonexpansive random operator. Then for each  $\omega \in \Omega$ ,*

$M(\omega) = \{x \in X: f(\omega, x) = x\}$  is nonempty, closed and convex and the mapping  $P: \Omega \times X \rightarrow X$  defined by  $P(\omega, x) = P_{M(\omega)}(x)$  is a nonexpansive random operator. Further, for any  $x \in X$ ,  $P(\cdot, x)$  is a random fixed point of  $f$ .

*Proof.* By a fixed point theorem of Browder [5], for each  $\omega \in \Omega$ ,  $M(\omega)$  is nonempty closed, and convex since a Hilbert space is strictly convex. Take any point  $v$  of  $X$  and a real sequence  $\{k_n\}$  as in the proof of Theorem 2.5. For each  $n$ , the  $(1 - k_n)$ -contraction random operator  $f_n: \Omega \times X \rightarrow X$  defined by  $f_n(\omega, x) = k_nv + (1 - k_n)f(\omega, x)$  has a unique random fixed point  $\xi_n$  by Hanš [12]. By Browder [6, Theorem 1], for any  $\omega \in \Omega$ ,  $\{\xi_n(\omega)\}$  converges to  $P(\omega, v)$ . Hence,  $P$  is a random operator (cf. Hille and Phillips [15, p. 74]). The last assertion is obvious. Q.E.D.

### 3. RANDOM FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

In this section we extend the results in Section 2 to multivalued mappings. The following is a random fixed point theorem for a multivalued condensing mapping. For a corresponding fixed point theorem, we refer to Petryshyn and Fitzpatrick [30].

**THEOREM 3.1.** *Let  $X$  be a separable closed convex subset of a Banach space,  $F: \Omega \times X \rightarrow CK(X)$  a continuous condensing random operator. Suppose that for any  $\omega \in \Omega$ ,  $F(\omega, X)$  is bounded. Then  $F$  has a random fixed point.*

*Proof.* Define a mapping  $G: \Omega \rightarrow 2^X$  by  $G(\omega) = \{x \in X: x \in F(\omega, x)\}$ , then by a fixed point theorem of Petryshyn and Fitzpatrick [30]  $G(\omega)$  is nonempty and compact for any  $\omega \in \Omega$ . Take a countable dense subset  $\{x_n\}$  of  $X$ . For each nonempty closed subset  $C$  of  $X$ , let

$$L(C) = \bigcap_{n=1}^{\infty} \bigcup_{x_i \in C_n} \{\omega \in \Omega: d(x_i, F(\omega, x_i)) < 2/n\},$$

where  $C_n = \{x \in X: d(x, C) < 1/n\}$ . We show that  $G^{-1}(C) = L(C)$ . If  $\omega \in G^{-1}(C)$ , then there is  $x \in C$  with  $x \in F(\omega, x)$ . Since  $F(\omega, \cdot)$  is continuous, it is continuous from  $X$  into the metric space  $(K(X), D)$  (cf. Castaing [8]). For each  $n$ , there exists  $x_{i(n)}$  such that  $\|x_{i(n)} - x\| < 1/n$  and  $D(F(\omega, x_{i(n)}), F(\omega, x)) < 1/n$ . We obtain

$$\begin{aligned} d(x_{i(n)}, F(\omega, x_{i(n)})) &\leq \|x_{i(n)} - x\| + d(x, F(\omega, x_{i(n)})) \\ &\leq \|x_{i(n)} - x\| + D(F(\omega, x), F(\omega, x_{i(n)})) \\ &< 1/n + 1/n = 2/n. \end{aligned}$$

As  $x_{i(n)} \in C_n$ , it follows that  $\omega \in L(C)$ . Conversely, if  $\omega \in L(C)$ , then for each  $n$ , we can take  $x_{i(n)} \in C_n$  for which  $d(x_{i(n)}, F(\omega, x_{i(n)})) < 2/n$ . Then the set  $B = \{x_{i(n)}\}$  is precompact by the same way as in the proof of [17, Proposition 1]. Without loss of generality, we may assume that  $\{x_{i(n)}\}$  itself converges to some  $x \in C$ . For each  $n$ , choose  $y_n \in F(\omega, x_{i(n)})$  such that  $\|x_{i(n)} - y_n\| = d(x_{i(n)}, F(\omega, x_{i(n)}))$ . Then  $\{y_n\}$  also converges to  $x$ . Since  $F(\omega, \cdot)$  is upper semicontinuous,  $x \in F(\omega, x)$ , hence  $\omega \in G^{-1}(C)$ . Therefore,  $G^{-1}(C) = L(C)$ . By Himmelberg [16],  $\{\omega \in \Omega: d(x, F(\omega, x)) < 2/n\} \in \mathcal{A}$  for any  $x \in X$ . Hence,  $G^{-1}(C) = L(C) \in \mathcal{A}$  and  $G$  is measurable. Any measurable selector of  $G$  is a random fixed point of  $F$ . Q.E.D.

We have the following corollary of Bohnenblust and Karlin type [4].

**COROLLARY 3.2.** *Let  $X$  be a compact (or separable and closed) convex subset of a Banach space,  $F: \Omega \times X \rightarrow CK(X)$  a continuous compact random operator. Then  $F$  has a random fixed point.*

**Remark 3.3.** If  $X$  is a separable bounded complete metric space, then any continuous condensing random operator  $F: \Omega \times X \rightarrow K(X)$ , where  $G(\omega)$  is not empty for every  $\omega \in \Omega$ , has a random fixed point by the same method as in the proof of Theorem 3.1.

Recall that a mapping  $F$  of a subset  $X$  of a Banach space  $E$  into  $CD(E)$  is called *demiclosed* if  $\{x_n\} \subset X$  and  $\{y_n\} \subset E$  with  $y_n \in F(x_n)$  are sequences such that  $\{x_n\}$  converges weakly to  $x_0$  and  $\{y_n\}$  converges to  $y_0$  in  $E$ , then  $y_0 \in F(x_0)$ .

Fixed point theorems corresponding to the following theorem were obtained by Lami Dozo [24] and Itoh and Takahashi [19].

**THEOREM 3.4.** *Let  $X$  be a weakly compact starshaped subset of a separable Banach space satisfying Opial's condition,  $F: \Omega \times X \rightarrow K(X)$  a nonexpansive random operator. Then  $F$  has a random fixed point.*

*Proof.* Choose a starcenter  $v$  of  $X$  and a sequence  $\{k_n\}$  of real numbers for which  $0 < k_n < 1$  and  $k_n \rightarrow 0$  as  $n \rightarrow \infty$ . For each  $n$ , define a  $(1 - k_n)$ -contraction random operator  $F_n: \Omega \times X \rightarrow K(X)$  by  $F_n(\omega, x) = k_n \bar{x} + (1 - k_n)F(\omega, x)$ , where  $y + B = \{y + z: z \in B\}$ , then by Itoh [18]  $F_n$  has a random fixed point  $\xi_n$ . For each  $n$ , define  $G_n: \Omega \rightarrow WK(X)$  by

$$G_n(\omega) = w\text{-cl}\{\xi_i(\omega): i \geq n\}.$$

Define  $G: \Omega \rightarrow WK(X)$  by

$$G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega),$$

then as in the proof of Theorem 2.5,  $G$  is  $w$ -measurable and has a measurable selector  $\xi$ .  $\xi$  is the desired random fixed point of  $F$ . Indeed, fix any  $\omega \in \Omega$ , then some subsequence  $\{\xi_m(\omega)\}$  of  $\{\xi_n(\omega)\}$  converges weakly to  $\xi(\omega)$ . For each  $m$ , there is an element  $u_m \in F(\omega, \xi_m(\omega))$  such that  $\xi_m(\omega) = k_m v + (1 - k_m) u_m$ . Since  $\xi_m(\omega) - u_m = k_m(v - u_m)$ ,  $\{\xi_m(\omega) - u_m\}$  converges to 0. By Lami Dozo [24],  $I - F(\omega, \cdot)$  is demiclosed. Thus, it follows that  $\xi(\omega) \in F(\omega, \xi(\omega))$ . Q.E.D.

In the sequel we give several common random fixed point theorems for singlevalued mappings and multivalued mappings. A mapping  $f: X \rightarrow X$  is said to *commute* with a mapping  $F: X \rightarrow CD(X)$  if for each  $x \in X$ ,  $f(F(x)) \subset F(f(x))$ . Also, a random operator  $f: \Omega \times X \rightarrow X$  is said to *commute* with a random operator  $F: \Omega \times X \rightarrow CD(X)$  if for each  $\omega \in \Omega$ ,  $f(\omega, \cdot)$  and  $F(\omega, \cdot)$  commute.

We refer to Itoh and Takahashi [19] for original results related to Theorems 3.5 and 3.6.

**THEOREM 3.5.** *Let  $X$  be a separable bounded closed convex subset of a Banach space,  $f: \Omega \times X \rightarrow X$  a continuous asymptotically regular random operator,  $F: \Omega \times X \rightarrow CK(X)$  a continuous condensing random operator. Suppose that  $f$  and  $F$  commute, then there exists a common random fixed point  $\xi$  of  $f$  and  $F$ , i.e., for each  $\omega \in \Omega$ ,  $f(\omega, \xi(\omega)) = \xi(\omega) \in F(\omega, \xi(\omega))$ .*

*Proof.* By Theorem 3.1,  $F$  has a random fixed point  $\xi_1$ . Then the mapping  $\xi_2: \Omega \rightarrow X$  defined by  $\xi_2(\omega) = f(\omega, \xi_1(\omega))$  is measurable by Himmelberg [16]. Since  $f$  and  $F$  commute,  $\xi_2$  is a random fixed point of  $F$ . By induction, the sequence  $\{\xi_n\}$  of mappings  $\xi_n: \Omega \rightarrow X$  for which  $\xi_{n+1}(\omega) = f(\omega, \xi_n(\omega))$  ( $\omega \in \Omega, n = 1, 2, 3, \dots$ ) are random fixed points of  $F$ . Since  $F$  is condensing, for any  $\omega \in \Omega$ ,  $\{\xi_n(\omega)\}$  is precompact. For each  $n$ , define  $G_n: \Omega \rightarrow K(X)$  by

$$G_n(\omega) = \text{cl}\{\xi_i(\omega): i \geq n\},$$

where  $\text{cl}(C)$  is the closure of  $C$ . Define  $G: \Omega \rightarrow K(X)$  by

$$G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega),$$

then  $G$  is measurable by Himmelberg [16], hence there exists a measurable selector  $\xi$  of  $G$ . This  $\xi$  is a common random fixed point of  $f$  and  $F$ . Indeed, for any fixed  $\omega \in \Omega$ , some subsequence  $\{\xi_m(\omega)\}$  of  $\{\xi_n(\omega)\}$  converges to  $\xi(\omega)$ . By the upper semicontinuity of  $F(\omega, \cdot)$ ,  $\xi(\omega) \in F(\omega, \xi(\omega))$ . Since  $f(\omega, \cdot)$  is asymptotically regular,  $\{\xi_{m+1}(\omega)\}$  also converges to  $\xi(\omega)$ . Hence  $f(\omega, \xi(\omega)) = \xi(\omega)$ . Q.E.D.

**THEOREM 3.6.** *Let  $X$  be a weakly compact starshaped subset of a separable Banach space which satisfies Opial's condition. Let  $f: \Omega \times X \rightarrow X$  be a nonexpan-*



sive and asymptotically regular random operator,  $F: \Omega \times X \rightarrow K(X)$  a nonexpansive random operator. Suppose that  $f$  commutes with  $F$ . Then there exists a common random fixed point of  $f$  and  $F$ .

*Proof.*  $F$  has a random fixed point  $\xi_1$  by Theorem 3.4. As in the proof of Theorem 3.5, there exists a sequence  $\{\xi_n\}$  of random fixed points of  $F$  such that  $\xi_{n+1}(\omega) = f(\omega, \xi_n(\omega))$  for  $\omega \in \Omega$ ,  $n = 1, 2, \dots$ . Define a sequence of mappings  $G_n: \Omega \rightarrow WK(X)$  and a mapping  $G: \Omega \rightarrow WK(X)$  by the same way as in the proof of Theorem 3.4. Then  $G$  is  $w$ -measurable and has a measurable selector  $\xi$ . We show that  $\xi$  is a common random fixed point of  $f$  and  $F$ . Fix any  $\omega \in \Omega$ , then some subsequence  $\{\xi_{m_i}(\omega)\}$  of  $\{\xi_n(\omega)\}$  converges weakly to  $\xi(\omega)$ . Since  $f(\omega, \cdot)$  is asymptotically regular,  $\{\xi_{m_i+1}(\omega)\}$  also converges weakly to  $\xi(\omega)$ . Since  $I - f(\omega, \cdot)$  and  $I - F(\omega, \cdot)$  are demiclosed (cf. Opial [28] and Lami Dozo [24]), it follows that  $f(\omega, \xi(\omega)) = \xi(\omega) \in F(\omega, \xi(\omega))$ . Q.E.D.

Now we prove a common random fixed point theorem for nonexpansive random operators in a Hilbert space. A fixed point theorem in this connection was given by Itoh and Takahashi [19].

**THEOREM 3.7.** *Let  $X$  be a bounded closed convex subset of a separable Hilbert space,  $f: \Omega \times X \rightarrow X$  a nonexpansive random operator,  $F: \Omega \times X \rightarrow CK(X)$  a nonexpansive random operator. Suppose that  $f$  and  $F$  commute. Then there exists a common random fixed point of  $f$  and  $F$ .*

*Proof.* Let  $P: \Omega \times X \rightarrow X$  be the nonexpansive random operator for  $f$  given in Theorem 2.6. Define a mapping  $G: \Omega \times X \rightarrow CK(X)$  by  $G(\omega, x) = F(\omega, P(\omega, x))$ , then by Itoh [18]  $G(\cdot, x)$  is measurable for every  $x \in X$ , hence  $G$  is a nonexpansive random operator. By Theorem 3.4,  $G$  has a random fixed point  $\zeta$ . For each fixed  $\omega \in \Omega$ ,

$$\begin{aligned} f(\omega, G(\omega, \zeta(\omega))) &= f(\omega, F(\omega, P(\omega, \zeta(\omega)))) \subset F(\omega, f(\omega, P(\omega, \zeta(\omega)))) \\ &= F(\omega, P(\omega, \zeta(\omega))) = G(\omega, \zeta(\omega)). \end{aligned}$$

Thus, by Itoh and Takahashi [19, Lemma],  $P(\omega, G(\omega, \zeta(\omega))) \subset G(\omega, \zeta(\omega))$ . In particular,  $P(\omega, \zeta(\omega)) \in G(\omega, \zeta(\omega))$ . It follows that the measurable mapping  $\xi: \Omega \rightarrow X$  defined by  $\xi(\omega) = P(\omega, \zeta(\omega))$  is a common random fixed point of  $f$  and  $F$ . Q.E.D.

When  $F$  is singlevalued, then we obtain the following

**COROLLARY 3.8.** *Let  $X$  be a bounded closed convex subset of a separable Hilbert space,  $f, g: \Omega \times X \rightarrow X$  nonexpansive random operators. Suppose that  $f$  and  $g$  commute, then there exists a common random fixed point of  $f$  and  $g$ .*

## 4. AN APPLICATION

In this section we give an application of Theorem 2.1 to a random differential equation in a Banach space. Let  $(\Omega, \mathcal{A})$  be a measurable space,  $E$  a separable real Banach space, and  $I$  a bounded closed interval of the real line  $R$ . Denote by  $C(I, E)$  the Banach space of all continuous mappings  $u: I \rightarrow E$  with the supremum norm  $\|u\|_\infty = \sup\{\|u(t)\|: t \in I\}$ . Let  $\mathcal{B}$  be the Borel field of  $C(I, E)$ ,  $\mathcal{B}_0$  the smallest  $\sigma$ -algebra of subsets of  $C(I, E)$  with respect to which the mappings  $P_t: u \rightarrow u(t)$  are measurable for all  $t \in I$ ,  $\mathcal{B}_1$  the smallest  $\sigma$ -algebra of subsets of  $C(I, E)$  with respect to which all continuous mappings  $g: C(I, E) \rightarrow E$  are measurable, respectively. It is known that if  $X$  is a metric space, then the Borel field of  $X$  is the smallest  $\sigma$ -algebra of subsets of  $X$  with respect to which all real valued continuous functions on  $X$  are measurable (cf. Parthasarathy [29, p. 4]). The following holds.

PROPOSITION 4.1.  $\mathcal{B} = \mathcal{B}_0 = \mathcal{B}_1$ .

*Proof.* We first show that  $\mathcal{B} \subset \mathcal{B}_0$ . Since  $C(I, E)$  is separable, every open set is a countable union of closed spheres and so it is sufficient to prove that  $\mathcal{B}_0$  contains all closed spheres in  $C(I, E)$ . Let  $\{t_n: n = 1, 2, \dots\}$  be the rational numbers in  $I$ , then for any  $r > 0$ ,  $u \in C(I, E)$ , we have

$$\{v \in C(I, E): \|v - u\|_\infty \leq r\} = \bigcap_{n=1}^{\infty} \{v \in C(I, E): \|v(t_n) - u(t_n)\| \leq r\}.$$

Since the intersection on the right side of this equation lies in  $\mathcal{B}_0$ , it follows that the sphere  $\{v \in C(I, E): \|v - u\|_\infty \leq r\} \in \mathcal{B}_0$ . Hence  $\mathcal{B} \subset \mathcal{B}_0$ . The inclusion that  $\mathcal{B}_0 \subset \mathcal{B}_1$  is obvious. Now we show that  $\mathcal{B}_1 \subset \mathcal{B}$ . Since  $E$  is separable,  $\mathcal{B}_1$  coincides with the smallest  $\sigma$ -algebra of subsets of  $C(I, E)$  with respect to which the mappings  $Q(x^*, g): u \rightarrow x^*(g(u))$  are measurable for all  $x^* \in E^*$  (the dual space of  $E$ ) and all continuous mappings  $g: C(I, E) \rightarrow E$  (cf. Bharucha-Reid [3, pp. 14–16] and Hille and Phillips [15, p. 72]). But  $Q(x^*, g)$  is a real valued continuous function on  $C(I, E)$ , thus  $\mathcal{B}_1 \subset \mathcal{B}$ . Therefore  $\mathcal{B} = \mathcal{B}_0 = \mathcal{B}_1$ .

Q.E.D.

Let  $\xi$  be a mapping of  $I \times \Omega$  into  $E$ .  $\xi$  is said to satisfy *condition*  $(C, \Omega)$  if for each  $\omega \in \Omega$ ,  $\xi(\cdot, \omega)$  is continuous and for each  $t \in I$ ,  $\xi(t, \cdot)$  is measurable. If  $\xi$  satisfies condition  $(C, \Omega)$ , then  $\xi$  is considered as a mapping of  $\Omega$  into  $C(I, E)$ . Concerning the measurability of  $\xi$ , we have the following characterization.

PROPOSITION 4.2.  $\xi$  satisfies condition  $(C, \Omega)$  if and only if  $\xi$  is measurable as a mapping of  $\Omega$  into  $C(I, E)$ .

*Proof.* Suppose that  $\xi$  satisfies condition  $(C, \Omega)$ . As in the proof of Proposition 4.1, it is enough to show that for any  $r > 0$ ,  $u \in C(I, E)$  and rational  $t \in I$ ,

the inverse image of the set  $B = \{v \in C(I, E) : \|v(t) - u(t)\| \leq r\}$  belongs to  $\mathcal{A}$ . Since  $\xi(t, \cdot)$  is measurable,

$$\begin{aligned}\xi^{-1}(B) &= \{\omega \in \Omega : \xi(\cdot, \omega) \in B\} \\ &= \{\omega \in \Omega : \|\xi(t, \omega) - u(t)\| \leq r\} \in \mathcal{A}.\end{aligned}$$

Conversely, suppose that  $\xi$  is measurable as a mapping of  $\Omega$  into  $C(I, E)$ . By Proposition 4.1, for any  $t \in I$ ,  $P_t$  is a measurable mapping of  $(C(I, E), \mathcal{B})$  into  $E$ , hence  $\xi(t, \cdot)$  is measurable. Thus  $\xi$  satisfies condition (C,  $\Omega$ ). Q.E.D.

Now we present a stochastic analogue of Ambrosetti [1, Teorema 3.1] as follows. Let  $x_0$  be an element of  $E$ ,  $B = \{x \in E : \|x - x_0\| \leq r\}$  and  $I = \{t \in R : |t - a| \leq T\}$ , where  $r, T > 0$ .

**THEOREM 4.3.** *Let  $f: I \times B \times \Omega \rightarrow E$  be a mapping having the following properties:*

- (i) *For each  $\omega \in \Omega$ ,  $f(\cdot, \cdot, \omega)$  is uniformly continuous on  $I \times B$ .*
- (ii) *For each  $\omega \in \Omega$ ,  $t \in I$ ,  $f(t, \cdot, \omega)$  is  $h(\omega)$ -set-Lipschitz, where  $\sup \{h(\omega) : \omega \in \Omega\} = h < +\infty$ .*
- (iii) *For each  $t \in I$ ,  $x \in B$ ,  $f(t, x, \cdot)$  is measurable.*
- (iv)  $m = \sup \{\|f(t, x, \omega)\| : t \in I, x \in B, \omega \in \Omega\} < +\infty$ .

*Let  $\eta: \Omega \rightarrow E$  be a measurable mapping for which  $\sup \{\|\eta(\omega) - x_0\| : \omega \in \Omega\} = r_0 < r$ . Denote  $I_1 = \{t \in R : |t - a| \leq T_1\}$ , where  $hT_0 < 1$  and  $T_1 = \min \{T, T_0, (r - r_0)/m\}$ . Then there exists a mapping  $\xi: I_1 \times \Omega \rightarrow E$  which satisfies condition (C,  $\Omega$ ) and that for any  $\omega \in \Omega$ ,*

$$\frac{d}{dt} \xi(t, \omega) = f(t, \xi(t, \omega), \omega) \quad (t \in I_1)$$

and  $\xi(a, \omega) = \eta(\omega)$ .

*Proof.* Let  $K = \{u \in C(I_1, E) : \|u(s) - u(t)\| \leq m |s - t| \text{ and } \|u(t) - x_0\| \leq r \text{ for all } s, t \in I_1\}$ , then  $K$  is a bounded closed convex subset of  $C(I_1, E)$ . Define a mapping  $W: \Omega \times K \rightarrow K$  by

$$W(\omega, u)(t) = \eta(\omega) + \int_a^t f(s, u(s), \omega) ds$$

( $\omega \in \Omega, u \in C(I_1, E), t \in I_1$ ). Then, for any  $s > t$ ,

$$\begin{aligned}\|W(\omega, u)(s) - W(\omega, u)(t)\| \\ \leq \int_t^s \|f(p, u(p), \omega)\| dp \leq m(s - t),\end{aligned}$$

and

$$\begin{aligned} & \|W(\omega, u)(t) - x_0\| \\ & \leq \|\eta(\omega) - x_0\| + \int_a^t \|f(s, u(s), \omega)\| ds \\ & \leq r_0 + mT_1 \leq r. \end{aligned}$$

Hence,  $W$  is well-defined. For any  $\omega \in \Omega$ ,  $W(\omega, \cdot)$  is continuous by (i), and for any  $u \in K$ ,  $W(\cdot, u)$  is measurable by (iii) and Proposition 4.2. By Ambrosetti [1], for any  $\omega \in \Omega$ ,  $W(\omega, \cdot)$  is a  $T_1 h(\omega)$ -set-contraction. Since  $T_1 h(\omega) \leq T_1 h < 1$ ,  $W(\omega, \cdot)$  is condensing. Thus, by Theorem 2.1 there exists a measurable mapping  $\xi: \Omega \rightarrow K$  such that for every  $\omega \in \Omega$ ,  $W(\omega, \xi(\omega)) = \xi(\omega)$ . By Proposition 4.2,  $\xi$  satisfies condition (C,  $\Omega$ ) as a mapping of  $I_1 \times \Omega$  into  $E$ . These implies that

$$\xi(t, \omega) = \eta(\omega) + \int_a^t f(s, \xi(s, \omega), \omega) ds$$

for all  $\omega \in \Omega$  and  $t \in I_1$ . This  $\xi$  is the desired solution.

Q.E.D.

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